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Simultaneous Best Approximation by Three Polynomials

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Let $[a, b]$ be any interval and let p_0, p_1, p_k be any three polynomials of degrees 0, 1, k, respectively, where $k \geqslant 2$. A set of necessary and sufficient conditions for the existence of an f in $C[a, b]$ such that p_i is the best approximation to f from the space of all polynomials of degree less than or equal to i, for all $i = 0, 1, k$, is given.

INTRODUCTION

T. J. Rivlin [8] posed the following problem:

Given polynomials $p_0, p_1, ..., p_{n-1}$ of degrees $0, 1, ..., n-1$, respectively, and an interval $[a, b]$, what are the necessary and sufficient conditions for the existence of f in $C[a, b]$ such that p_i is the best approximation to f from $\mathscr{P}_i$, for all $i = 0, 1, 2, ..., n-1$.

Rivlin [8] has shown that for that to be true, it is necessary that for all i, j in $0, 1, ..., n-1$, the polynomial $p_i - p_j$ is either zero or changes sign at least i times in $[a, b]$.

Deutsch, Morris, and Singer [4] have, among more general results, proved that the above necessary condition of Rivlin is also sufficient for $n = 2$.

Sprecher [9, 10] has extended this result to the case of two polynomials of arbitrary degrees and proved that this condition is not sufficient for $n = 3$. Further, he has given a solution to the above problem in the case $n = 3$.

Subrahmanya [14, 12] has given a solution to the above problem for a general n. References [4, 9–12] have considered this problem in more general settings.

The main result of this paper yields, as a particular case, another solu-

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tion to Rivlin's problem in the case $n=3$. At the end of this paper, we describe many aspects in which our solution is better than that of Sprecher. Here it suffices to remark that this paper, for the first time, solves the problem in the more general case (p_0, p_1, p_k) for higher values of k.

Some Notations. For a nonnegative integer i, $\mathscr{P}_i=$ the space of all polynomials of degree i.

$p_i=$ a polynomial of degree i.

Let p_0, p_1, p_k be given, where $k\geqslant 2$. We say that (p_0, p_1, p_k) is admissible on an interval $[a, b]$, if there is f in $C[a, b]$ such that p_i is the best approximation to f from the subspace $\mathscr{P}_i$ of $C[a, b]$, for $i=0$, 1, and k.

We consider the problem: When is a given (p_0, p_1, p_k) admissible on a given $[a, b]$? For $k=2$ alone, it has been considered by others.

1. The Main Theorem

We need the following lemmas. Since they are not found anywhere in this form, we include their proofs for the sake of completeness.

Lemma 1. *Let f, g in $C[a, b]$ be such that $f\leqslant g$. Then for any $x_1, x_2, ..., x_n$ in $[a, b]$ and real numbers $y_1, y_2, ..., y_n$ such that $f(x_i)\leqslant y_i\leqslant g(x_i)$, there exists h in $C[a, b]$ such that $f\leqslant h\leqslant g$ and $h(x_i)=y_i$ for all $i=1, 2, ..., n$.*

Proof. Choose $0\leqslant\lambda_i\leqslant 1$ such that

$$y_i=\lambda_i f(x_i)+(1-\lambda_i)\, g(x_i), \qquad i=1, 2, ..., n.$$

Choose a continuous $\varphi\colon [a, b]\to[0, 1]$ such that

$$\varphi(x_i)=\lambda_i, \qquad i=1, 2, ..., n.$$

Define $h\colon [a, b]\to\mathbb{R}$ by

$$h(x)=\varphi(x) f(x)+(1-\varphi(x))\, g(x).$$

Then h is as required in the lemma.

Lemma 2. *Let $f\colon [a, b]\to\mathbb{R}$ be a strictly increasing or strictly decreasing function, Let*

$$a\leqslant y_1<x_1<y_2<x_2<y_3\leqslant b.$$

Then

(i) *either* $|f(x_1)|$ *or* $|f(x_2)|$ *is* $<\max\{f(y_1), -f(y_2), f(y_3)\}$.

(ii) *either* $|f(x_1)|$ *or* $|f(x_2)|$ *is* $<\max\{-f(y_1), f(y_2)-f(y_3)\}$.

Proof. (i) *Case* 1: f is strictly increasing.
If $f(x_2)\geqslant 0$, then $|f(x_2)|=f(x_2)<f(y_3)$.
If $f(x_2)\leqslant 0$, then $|f(x_2)|=-f(x_2)<-f(y_2)$.

Case 2: f is strictly decreasing.

If $f(x_1)\geqslant 0$, then $|f(x_1)|=f(x_1)<f(y_1)$.
If $f(x_1)\leqslant 0$, then $|f(x_1)|=-f(x_1)<-f(y_2)$.

Similarly, we can prove (ii).

THEOREM 1. (p_0, p_1, p_k) *is admissible on* $[a, b]$ *if and only if there exist five points*

$$a\leqslant t_{01}\leqslant t_{02}<b,$$

$$a\leqslant t_{11}\leqslant t_{12}<t_{13}\leqslant b,$$

and numbers η_0, η_1 in $\{-1, 1\}$ *such that*

$$\text{(i)}\quad \min_{\substack{j=1,2\\ i=1,k}} \eta_0(-1)^j\,[(p_i-p_0)(t_{0j})]>0$$

$$\text{(ii)}\quad \min_{1\leqslant s\leqslant 3} \eta_1(-1)^s\,[(p_k-p_1)(t_{1s})]>0$$

and

$$\text{(iii)}\quad \max_{1\leqslant s\leqslant 3} \eta_1(-1)^s\,[(p_1-p_0)(t_{1s})]<\min_{\substack{j=1,2\\ i=1,k}} \eta_0(-1)^j\,[(p_i-p_0)(t_{0j})].$$

Remark. This statement will be discussed in the next section.

Proof. For the sake of convenience in the proof, we shall make use of the notations

$$C_{i,0}=\min_{j=1,2} \eta_0(-1)^j\,[(p_i-p_0)(t_{0j})]$$

$$C_{i,1}=\min_{1\leqslant s\leqslant 3} \eta_1(-1)^s\,[(p_i-p_1)(t_{1s})]$$

for $i=0, 1, k$. Then note that the inequalities take the form

(i) $C_{i,0}>0$ for $i=1, k$,

(ii) $C_{k,1}>0$,

(iii) $-C_{0,1}<C_{i,0}$ for $i=1, k$.

Necessity. Suppose (p_0, p_1, p_k) is admissible on $[a, b]$. Let f be an element in $C[a, b]$ for which p_0, p_1, p_k are the best approximations, respectively, from $\mathscr{P}_0, \mathscr{P}_1, \mathscr{P}_k$. Let $e_i = \|f - p_i\|$ for $i = 0, 1, k$. Then by the alternation theorem [3] there exist points

$$\begin{aligned} &a \leqslant t_{01} < t_{02} \leqslant b \\ &a \leqslant t_{11} < t_{12} < t_{13} \leqslant b \end{aligned} \tag{T}$$

and numbers η_0, η_1 in $\{-1, 1\}$ such that

$$\eta_0(-1)^j [f - p_0)(t_{0j})] = e_0, \qquad \text{for} \quad j = 1, 2$$

and

$$\eta_1(-1)^s [(f - p_i)(t_{1s})] = e_1, \qquad \text{for} \quad s = 1, 2, 3.$$

Now

$$\begin{aligned} &\eta_0(-1)^j [(p_i - p_0)(t_{0j})] \\ &\quad = \eta_0(-1)^j [(f - p_0)(t_{0j}) - (f - p_i)(t_{0j})] \\ &\quad \geqslant e_0 - e_i \qquad \text{because } (f - p_i)(t_{0j}) \leqslant \|f - p_i\| = e_i. \end{aligned}$$

In our notation, this proves

$$C_{i,0} \geqslant e_0 - e_i \qquad \text{for} \quad i = 1, 2, k.$$

Since $e_0 > e_1 > e_k$, (i) follows from the above. Similarly, one can prove that

$$C_{i,1} \geqslant e_1 - e_i \qquad \text{for} \quad i = 1, 2, k$$

and

$$C_{0,1} \geqslant e_1 - e_0.$$

Part (ii) follows from the former of these two inequalities. Now we have

$$-C_{0,1} \leqslant e_0 - e_1 \leqslant C_{1,0}$$

and

$$-C_{0,1} \leqslant e_0 - e_1 < e_0 - e_k \leqslant C_{k,0}.$$

By the continuity of the functions $p_i - p_j$ and by the fact that $p_1 - p_0$ has a straight-line graph, we can choose points of T such that the above inequalities are strict, when computed with respect to the new set of five points.

Sufficiency. We can assume without loss of generality that all the t_{ij}'s in the hypothesis are distinct.

Let x_0 be the root of $p_0 - p_1$. Then from (i), $a < x_0 < b$. From (ii) we find that there exist two roots x_1, x_2 of $p_1 - p_k$ such that $a \leqslant t_{11} < x_1 < t_{12} < x_2 < t_{13} \leqslant b$. Therefore by Lemma 2, for every η in $\{-1, 1\}$ there exists $1 \leqslant i \leqslant 2$ such that

$$|(p_1 - p_0)(x_i)| < \max_{1 \leqslant s \leqslant 3} \eta(-1)^s [(p_1 - p_0)(t_{1s})]$$
$$= -C_{0,1} \qquad \text{when} \quad \eta = \eta_1.$$

Choose a number $e_{01} > 0$ such that $-C_{0,1} < e_{01} < \min\{C_{1,0}, C_{k,0}\}$. This is possible because of (i) and (iii).

Let $C = \frac{1}{2}[\min\{C_{1,0}, C_{k,0}\} - e_{01}] > 0$. There is $\varepsilon_1 > 0$ such that

$$|(p_1 - p_0)(t)| < e_{01} \qquad \text{for all } t \text{ in } [x_i - \varepsilon_1, x_i + \varepsilon_1].$$

Since x_i is a root of $p_k - p_1$, there is $\varepsilon_2 > 0$ such that

$$|(p_k - p_1)(t)| < \min\{C_{k,1}, C\} \qquad \text{for all } t \text{ in } [x_i - \varepsilon_2, x_i + \varepsilon_2].$$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Choose any $k+2$ points

$$t_{k1} < t_{k2} < \cdots < t_{k,k+2} \qquad \text{in} \quad [x_i - \varepsilon, x_i + \varepsilon].$$

Then

$$(-1)^l [(p_k - p_1)(t_{kl}] \leqslant (p_k - p_1)(t_{kl})$$
$$< \min\{C_{k,1}, C\} \qquad \text{for} \quad 1 \leqslant l \leqslant k+2. \tag{1}$$

Choose any number $e_{1k} > 0$ such that

$$\max_{1 \leqslant l \leqslant k+2} (-1)^l [(p_k - p_1)(t_{kl})] < e_{1k} < \min\{C_{k,1}, C\}. \tag{2}$$

Now

$$C_{k,0} > C + e_{01}$$
$$> e_{1k} + e_{01}$$
$$> (-1)^l [(p_k - p_0)(t_{kl})] + (-1)^l [(p_1 - p_0)(t_{kl})]$$
$$(\text{since } t_{kl} \in [x_i - \varepsilon_1, x_i + \varepsilon_1])$$
$$= (-1)^l [(p_k - p_0)(t_{kl})] \qquad \text{for all } 1 \leqslant l \leqslant k+2.$$

Denote $e_{01}+e_{1k}$ by e_{0k}. Then

$$(-1)^l[(p_k-p_0)(t_{kl})]<e_{0k}<C_{k,0} \qquad \text{for all } 1\leqslant l\leqslant k+2. \tag{3}$$

Choose a number r_{1k} such that $r_{1k}>\max_{i,j=0,1,k}\|p_i-p_j\|$. Define

$$r_{0k}=r_{1k}+e_{01}$$
$$r_{01}=r_{0k}+e_{1k}$$
$$e_0=(r_{01}+r_{0k}-r_{1k})/2$$
$$e_1=(r_{01}+r_{1k}-r_{0k})/2$$

and

$$e_k=(r_{0k}+r_{1k}-r_{01})/2.$$

Clearly $e_0>e_1>e_k$.

Now e_k is found to be $=(r_{1k}-e_{1k})/2>0$ because $e_{1k}<\|p_1-p_k\|<r_{1k}$. We also see that

$$e_0-e_1=r_{0k}-r_{1k}=e_{01}$$
$$e_0-e_k=r_{01}-r_{1k}=e_{0k}$$
$$e_1-e_k=r_{01}-r_{0k}=e_{1k}$$
$$e_0+e_1=r_{01}$$
$$e_0+e_k=r_{0k}$$
$$e_1+e_k=r_{1k}.$$

From (1), (2), (3) and the fact $\|p_i-p_j\|<e_i+e_j$, one can prove

$$\max_{i=0,1,k}(p_i(t_{0j})-e_i)\leqslant p_0(t_{0j})+\eta_0(-1)^j e_0 \leqslant \min_{h=0,1,k}(p_h(t_{0j})+e_h) \qquad \text{for all } j=1,2$$

$$\max_{i=0,1,k}(p_i(t_{1s})-e_i)\leqslant p_i(t_{1s})+\eta_1(-1)^s e_1 \leqslant \min_{h=0,1,k}(p_h(t_{1s})+e_s) \qquad \text{for all } s=1,2,3$$

and

$$\max_{i=0,1,k}(p_i(t_{kl})-e_i)\leqslant p_k(t_{kl})+(-1)^l e_k \leqslant \min_{h=0,1,k}(p_h(t_{kl})+e_h \qquad \text{for all} \quad 1\leqslant l\leqslant k+2.$$

By Lemma 1, there exists f in $C[a, b]$ such that

$$\max_{i=0,1,k} (p_i(t)-e_i) \leqslant f(t) \leqslant \min_{h=0,1,k} (p_h(t)+e_h) \qquad \text{for all} \quad a \leqslant t \leqslant b$$

and

$$f(t_{0j}) = p_0(t_{0j}) + \eta_0(-1)^j e_0, \qquad 1 \leqslant j \leqslant 2$$

$$f(t_{1s}) = p_1(t_{1s}) + \eta_1(-1)^s e_1, \qquad 1 \leqslant s \leqslant 3$$

and

$$f(t_{k1}) = p_k(t_{k1}) + (-1)^l e_k, \qquad 1 \leqslant 1 \leqslant k+2.$$

It follows from the alternation theorem [3] that p_i is the best approximation to this f from $\mathscr{P}_i$ for $i=0, 1, k$.

3. Discussion on the Theorem

Remark 1. This theorem is better than Sprecher's [10] theorem for the following reasons:

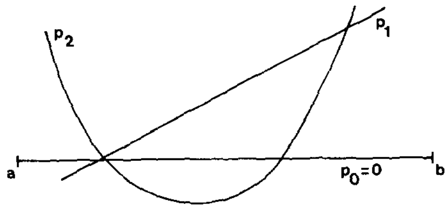
(1) This solves the problem relating to more general triples of polynomials (p_0, p_1, p_k). Sprecher's solution is valid only for (p_0, p_1, p_2), that is, only when $k=2$. In other words, our answer to Rivlin's problem in the case $n=3$ is in a form that can be adopted for more general situations.

(2) Our solution is a natural extension of Rivlin's necessary condition. The conditions (i) and (ii) of this theorem subsume Rivlin's conditions, as can be easily proved.

(3) The reformulation of the inequalities in the theorem, as stated in the beginning of the proof, is noteworthy. It is in the form: The five numbers $C_{0,1}$, $C_{0,k}$, $C_{k,1}$, $C_{0,1}+C_{1,0}$, and $C_{k,1}+C_{1,0}$ are all positive. This not only makes the theorem easy to remember, but also paves the way for solving Rivlin's problem for higher values of n. It may be noted that, after solving for $n=3$, Sprecher [4] has remarked that for $n=4$ onwards, the solution cannot be as easy. Contrary to this, our solution gives an insight that has been exploited in [7].

Moreover, a geometric solution to Rivlin's problem for $n=3$, which can be easily visualised, can be deduced (see Remark 5).

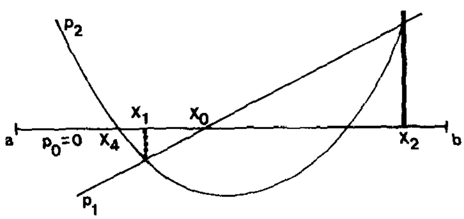
Remark 2. Let us take $k=2$. The condition (i) of the theorem states that there exist points $a \leqslant s < t \leqslant b$ such that $p_0(s) < p_i(s)$ and $p_0(t) > p_i(t)$ for $i=1, 2$. Rivlin's condition states that $p_1 - p_0$ changes sign at least once, $p_2 - p_0$ changes sign at least once, and $p_2 - p_1$ changes sign at least twice, in the interval $[a, b]$. The following is a simple example that satisfies Rivlin's conditions, but not our condition (i):



SCHEME 1.

In this example, there is no t in $[a, b]$ such that $p_0(t) > p_i(t)$ for both $i = 1, 2$. In fact, this is an example of (p_0, p_1, p_2) that is not admissible on any interval $[x, y]$.

Remark 3. We now give an example that satisfies (i) and (ii) but not (iii):



SCHEME 2.

We note that the points t_{1k}, $1 \leqslant k \leqslant 3$, must be chosen such that $a \leqslant t_{11} < x_1 < t_{12} < x_2 < t_{13} \leqslant b$. But on $[a, x_1]$, $p_2 > p_1$. Hence $\eta_1 = -1$.

Also t_{01} and t_{02} are in $[a, b]$ at which $p_0 > p_i$ and $p_0 < p_i$ respectively for both $i = 1, 2$. Therefore t_{01} must be chosen between x_4 and x_0 (see the diagram). Now

$$\max_{i=1,2} \| p_0 - p_i \|_{[x_4, x_0]} < (p_1 - p_0)(x_2).$$

Hence we cannot choose t_{13} in $(x_2, b]$ such that

$$(p_1 - p_0)(t_{13}) < (p_0 - p_i)(t), \qquad i = 1, 2,$$

for some $t \in [a, b]$ such that $p_0(t) > p_i(t)$, $i = 1, 2$.

Hence in this example, (i), (ii) are satisfied, but not (iii). To visualise why (iii) fails here, we note that the dotted vertical segment is shorter than the thick vertical segment. This should not happen if (p_0, p_1, p_2) is admissible.

Remark 4. Let p_0, p_1, p_k with $2 \leqslant k \leqslant n$ be given. Then there exists f in $C[a, b]$ such that p_0, p_1, p_k are the best approximations to f from $\mathscr{P}_0, \mathscr{P}_1, \mathscr{P}_n$ if and only if there exists g in $C[a, b]$ such that p_0, p_1, p_k are the best approximations to g from $\mathscr{P}_0, \mathscr{P}_1, \mathscr{P}_k$. This is easily seen from the proof of our theorem.

Remark 5. In [7], the following geometrical result has also been deduced from the theorem of Section 1.

THEOREM. *(p_0, p_1, p_2) is admissible on some interval if and only if*

$$\| p_0 - p_2 \|_{I \setminus J} > \| p_0 - p_2 \|_{J \setminus I},$$

where $I =$ the interval with end points the roots of $p_2 - p_0$ and $J =$ the interval with end points the roots of $p_2 - p_1$, are nondegenerate intervals.

This theorem yields as a corollary a geometrical solution to Rivlin's problem in the case $n = 3$.

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